Dual Nature of Input-Output Modeling:
the Matrix-Valued Cost and Production Functions

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A general linear problem of input-output analysis is considered in the paper as a system of equations written in terms of free variables for any rectangular input-output table given. This system spans the regular linear equations for material and financial balances, a batch of predetermined values for exogenous variables (in turn, net final demand and gross value added) and an additional set of linkage equations that provides the exact identifiability for all unknown variables.

Variations in exogenous elements of input-output model lead to the changes of price and quantity proportions in the resulting production and intermediate consumption matrices that are formally described by two nonlinear multiplicative patterns. It is shown how these patterns can be linearized and adjusted for evaluating the input-output model at constant prices and at constant level of production.

The strict identifiability of input-output model at constant prices is achieved by introducing to the system of its equations either (1) the linear matrix-valued *cost* function with industry outputs as its arguments based on (Leontief) technical coefficients or (2) the linear matrix-valued *production* function with industry inputs as its arguments based on industry productive (quasi-reciprocal technical) coefficients. In contrast, the model at constant level of production is exactly identifiable provided that one involves in it either (3) the linear matrix-valued *cost* function with product outputs as its arguments based on (Ghosh) allocation coefficients or (4) the linear matrix-valued *production* function with product inputs as its arguments based on product multiplication (quasi-reciprocal allocation) coefficients.

Identification of the production and intermediate consumption matrices at constant prices and at constant production level for rectangular input-output tables leads to the pair of trivial model solutions with exogenous value added and exogenous final demand, respectively. Nevertheless, for square input-output tables there are also the pair of nontrivial supplementary solutions with exogenous final demand at constant prices and exogenous value added at constant level of production. It is important to emphasize here that using matrix-valued production functions (2) and (4) with quasi-reciprocal technical and allocation coefficients gives the same solutions as introducing matrix-valued cost functions (1) and (3) with conventional coefficients, respectively. Thus, technical and allocation coefficients should be regarded as helpful ways of economic interpretation rather than as basic framework or operational tools for modeling. Moreover, equivalence of the models with the matrix-valued production functions and the models with the matrix-valued cost functions can be appreciated as a clear demonstration of general equilibrium in the theory of input-output analysis and an ostensive evidence of dual nature of input-output modeling.

Obtained supplementary solutions are used for formulating the advanced versions of Leontief demand-driven model and Ghosh supply-driven model with generalized technical and allocation coefficients. For a symmetric input-output table with diagonal production matrix, the generalized demand-driven and supply-driven models can be easily transformed to the classical forms of Leontief and Ghosh input-output models. The equivalence of Leontief price model and Ghosh supply-driven model as well as the equivalence of Leontief demand-driven model and Ghosh quantity model is proven.

Keywords: rectangular input-output table, matrix-valued production and cost functions, exogenous changes in final demand and value added, demand-driven and supply-driven models, quantity and price models

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1. Linear input-output model: a general formulation

The general linear input-output model of an economy with N products (commodities) and M industries (sectors) for the certain time period leans on a pair of rectangular matrices, namely supply (production) matrix $\mathbf{X}$ and use for intermediates (intermediate consumption) matrix $\mathbf{Z}$ of the same dimension $N\times M$ both. In mathematical notation, the model includes the vector equation for material balance of products' intermediate and final uses, i.e.,

$$\mathbf{X}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y}\,, \tag{1}$$

and the following vector equation for financial balance of industries' intermediate and primary (combined into value added) inputs:

$$\mathbf{e}'_N\mathbf{X} = \mathbf{e}'_N\mathbf{Z} + \mathbf{v}' \tag{2}$$

where $\mathbf{e}_N$ and $\mathbf{e}_M$ are N×1 and M×1 summation column vectors with unit elements, $\mathbf{y}$ is a column vector of net final demand with dimensions N×1, and $\mathbf{v}$ is a column vector of value added with dimensions M×1. Here putting a prime after vector's (matrix's) symbol denotes a transpose of this vector (matrix).

"One of the major uses of the information in an input-output model is to assess the effect on an economy of changes in elements that are exogenous to the model of that economy" (Miller and Blair, 2009, p. 243). To measure the changes mentioned above, in most practical cases there usually is the supply and use table for economy under consideration for some time period (say, period 0) compiled from available statistical data. This table includes the production matrix X_0 and intermediate consumption matrix $\mathbf{Z}_0$ with dimensions N×M, (N×1)-dimensional column vector of net final demand $\mathbf{y}_0$, and (M×1)-dimensional column vector of value added $\mathbf{v}_0$ (see Eurostat, 2008). Note that the equations (1) and (2) are exactly met for the initial supply and use table components.

With accordance to the quotation above, the main aim of constructing input-output models is to assess an impact of the exogenous changes (either absolute or relative) in net final demand and, by virtue of symmetry in the balance equations under consideration, an impact of the exogenous changes in gross value added on simultaneous behavior of the economy as a whole and its industries. Balance models do not usually reflect the true causes of the certain changes in final demand or value added, so the response of the economy to any exogenous disturbance of some model component is evaluated in the mode of getting answers to the questions like "what would happen if ...? ".

The balance model (1), (2) contains N+M linear equations with 3(N+M) scalar variables. Assume that exogenous disturbance is expressed in terms of k exogenous variables. To provide exact (or strict) identifiability of the model it is required to incorporate into it $2(N+M) - k$ auxiliary independent equations as a certain set of linkages between the variables. In particular, $N+2M$ independent equations are needed at $k = N$, and $2N+M$ equations are needed at $k = M$. The structure of initial supply and use table serves as an informational framework for constructing the auxiliary linkage equations.

2. The price and quantity transformations of the model variables

In principle, any finite variations in exogenous elements of the input-output model (1), (2) lead to the changes of price and quantity proportions in the resulting (i.e., disturbed) supply and use table. The most general way to describe an impact of these changes on matrices **X** and **Z** is as follows:

$$\mathbf{X} = \mathbf{P}_{\mathbf{X}} \circ \mathbf{Q}_{\mathbf{X}} \circ \mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{P}_{\mathbf{Z}} \circ \mathbf{Q}_{\mathbf{Z}} \circ \mathbf{Z}_0$$

where $\mathbf{P}_{\mathbf{X}}$ and $\mathbf{P}_{\mathbf{Z}}$ are $N\times M$-dimensional matrices of the relative price indices for products, $\mathbf{Q}_{\mathbf{X}}$ and $\mathbf{Q}_{\mathbf{Z}}$ are $N\times M$ matrices of the relative quantity (physical volume) indices for industries of the economy, and the character " $\circ$ " denotes the Hadamard's (element-wise) product of two matrices with the same dimensions.

Following Motorin (2017), one can assume that in market economy $\mathbf{P}_{\mathbf{X}} = \mathbf{P}_{\mathbf{Z}} = \mathbf{P}$, and $\mathbf{Q}_{\mathbf{X}} = \mathbf{Q}_{\mathbf{Z}} = \mathbf{Q}$ on the current level of production. Besides, it is quite natural to propose also that the price on certain product does not vary along the row of producing-and-consuming industries, i.e., $p_{nm} = p_n$ for all $m = 1 \div M$ at $n = 1 \div N$ where the character " $\div$ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range, and, moreover, that the production quantity index for the certain industry's output and intermediate consumption is keeping invariable through all products produced and consumed, namely, $q_{nm} = q_m$ for all $n = 1 \div N$ at $m = 1 \div M$.

Thus, matrices **P** and **Q** can be represented respectively as $\mathbf{P} = \mathbf{p} \otimes \mathbf{e}'_M$ and $\mathbf{Q} = \mathbf{e}_N \otimes \mathbf{q}'$ where **p** is a column vector of the relative price indices on products with dimensions $N\times 1$, **q** is a column vector of the relative quantity indices for industries with dimensions $M\times 1$, and the character " $\otimes$ " denotes the Kronecker product for two matrices.

Transforming the above statements into regular matrix notation gives two nonlinear multiplicative patterns

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0\hat{\mathbf{q}} \qquad (3)$$

where putting a "hat" over vector's symbol (or angled bracketing around it) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697). The patterns (3) provide the combined price and quantity description of an economy response to exogenous changes in the input-output model's variables, inter alia, in net final demand and in gross value added.

The nonlinear multiplicative patterns (3) generate a nonlinear problem of input-output analysis as follows:

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{q} = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{q} + \mathbf{y}, \qquad \mathbf{p}\mathbf{X}_0\hat{\mathbf{q}} = \mathbf{p}\mathbf{Z}_0\hat{\mathbf{q}} + \mathbf{v}'.$$

Note that here the unknown vectors $\mathbf{p}$ and $\mathbf{q}$ cannot be estimated unambiguously because the patterns (3) are hyperbolically homogeneous, since $\mathbf{X} = \mathbf{pq}' \circ \mathbf{X}_0$, $\mathbf{Z} = \mathbf{pq}' \circ \mathbf{Z}_0$, and $\mathbf{pq}' = c\mathbf{p} \cdot \mathbf{q}'/c$ for any nonzero scalar *c*.

Nevertheless, evaluating of input-output model (1), (2) in terms of the production quantity changing at constant prices on the products and/or in terms of price changing at constant level of production in the industries is of great theoretical and practical interest.

3. The linear input-output models at constant prices and at constant production level

In a case of constant prices on products we have $\hat{\mathbf{p}} = \mathbf{E}_N$ where $\mathbf{E}_N$ is identity matrix of order *N*, so the nonlinear multiplicative patterns (3) can be rewritten in linear form, namely

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}. \tag{4}$$

Substituting multiplicative patterns (4) in the equations of input-output model (1), (2), we obtain

$$(\mathbf{X}_0 - \mathbf{Z}_0)\,\mathbf{q} = \mathbf{y}, \tag{5}$$

$$\langle \mathbf{e}'_N (\mathbf{X}_0 - \mathbf{Z}_0)\rangle \mathbf{q} = \mathbf{v} \tag{6}$$

respectively.

Assessing the input-output model (1), (2) at constant level of production in the industries (at $\hat{\mathbf{q}} = \mathbf{E}_M$ where $\mathbf{E}_M$ is identity matrix of order *M*) leads to following linear patterns

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0. \tag{7}$$

Finally, substituting multiplicative patterns (7) in the equations of input-output model (1), (2), we have

$$\langle (\mathbf{X}_0 - \mathbf{Z}_0)\,\mathbf{e}_M \rangle \mathbf{p} = \mathbf{y}, \tag{8}$$

$$(\mathbf{X}'_0 - \mathbf{Z}'_0)\,\mathbf{p} = \mathbf{v} \tag{9}$$

respectively.

4. Exploring an input-output model at constant prices

According to the ***first*** equation (4), the row vector of industry outputs is equal to $\mathbf{e}'_N\mathbf{X} = \mathbf{e}'_N\mathbf{X}_0\hat{\mathbf{q}} = \mathbf{q}'\langle \mathbf{e}'_N\mathbf{X}_0\rangle$ from which the quantity index $\mathbf{q}$ follows as $\hat{\mathbf{q}} = \langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{X}\rangle$ where the obvious commutativity property of diagonal matrices is used. Substituting the latter expression in multiplicative patterns (4) gives two linear matrix-valued functions

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}} = \mathbf{X}_0\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{X}\rangle = \mathbf{G}\langle \mathbf{e}'_N\mathbf{X}\rangle, \tag{10}$$

$$\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}} = \mathbf{Z}_0\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{X}\rangle = \mathbf{A}\langle \mathbf{e}'_N\mathbf{X}\rangle \tag{11}$$

with vector of industry outputs $\mathbf{e}'_N\mathbf{X}$ as their common argument. Note that matrix $\mathbf{G} = \mathbf{X}_0\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}$ is known in special literature as product-mix matrix (see Eurostat, 2008) with shares of each product in output of an industry in a column. The matrix **G** in (10) provides a linkage between production matrix **X** and its column marginal totals.

In turn, matrix-valued function (11) establishes a linear dependency of intermediate consumption matrix **Z** from the industry outputs $\mathbf{e}'_N\mathbf{X}$, and so it can be classified as *the linear matrix-valued cost function*. Matrix $\mathbf{A} = \mathbf{Z}_0\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}$ is widely known under the name of (Leontief) technical coefficients matrix (see, e.g., Miller and Blair, 2009).

Substituting the matrix-valued cost function (11) in the equations of input-output model (1), (2), we have

$$\mathbf{X}\mathbf{e}_M = \mathbf{A}\mathbf{X}'\mathbf{e}_N + \mathbf{y}\,, \qquad \mathbf{e}'_N\mathbf{X} = \mathbf{e}'_N\mathbf{A}\langle \mathbf{e}'_N\mathbf{X}\rangle + \mathbf{v}'\,.$$

Proceeding by using formula (10) gives

$$\mathbf{G}\mathbf{X}'\mathbf{e}_N = \mathbf{A}\mathbf{X}'\mathbf{e}_N + \mathbf{y}\,, \qquad \mathbf{X}'\mathbf{e}_N = \langle \mathbf{e}'_N\mathbf{A}\rangle\mathbf{X}'\mathbf{e}_N + \mathbf{v}\,,$$

and finally we obtain system of equations

$$(\mathbf{X}_0 - \mathbf{Z}_0)\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'\mathbf{e}_N = \mathbf{y}\,, \qquad \langle \mathbf{e}'_N(\mathbf{X}_0 - \mathbf{Z}_0)\rangle\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'\mathbf{e}_N = \mathbf{v}$$

that exactly corresponds to the system (5), (6) provided that $\hat{\mathbf{q}} = \langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{X}\rangle$.

From the other side, in accordance with the ***second*** equation (4) the row vector of industry expenditures for intermediate consumption is equal to $\mathbf{e}'_N\mathbf{Z} = \mathbf{e}'_N\mathbf{Z}_0\hat{\mathbf{q}} = \mathbf{q}'\langle \mathbf{e}'_N\mathbf{Z}_0\rangle$ from which index **q** follows as $\hat{\mathbf{q}} = \langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{Z}\rangle$. Substituting the latter expression in multiplicative patterns (4) yields two linear matrix-valued functions

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}} = \mathbf{X}_0\langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{Z}\rangle = \tilde{\mathbf{A}}\langle \mathbf{e}'_N\mathbf{Z}\rangle\,, \tag{12}$$

$$\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}} = \mathbf{Z}_0\langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{Z}\rangle = \tilde{\mathbf{G}}\langle \mathbf{e}'_N\mathbf{Z}\rangle \tag{13}$$

with vector of industry intermediate inputs $\mathbf{e}'_N\mathbf{Z}$ as their common argument. Here matrix-valued function (12) provides a linear dependency of output matrix **X** from the industry intermediate inputs $\mathbf{e}'_N\mathbf{Z}$, and so it can be classified as *the linear matrix-valued production function*. Matrix $\tilde{\mathbf{A}} = \mathbf{X}_0\langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}$ is apparently not known in special literature in contrast to Leontief technical coefficients matrix $\mathbf{A} = \mathbf{Z}_0\langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}$, but it is easy to see that matrices $\tilde{\mathbf{A}}$ and **A** are in certain "quasi-reciprocal" relation.

The equation (13) establishes a linkage between intermediate consumption matrix **Z** and its column marginal totals. Matrix $\tilde{\mathbf{G}} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1}$ are apparently not mention in special literature in contrast to its well-known twin – the product-mix matrix $\mathbf{G} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$.

Substituting the matrix-valued production function (12) in the equations of input-output model (1), (2), we obtain

$$\tilde{\mathbf{A}}\mathbf{Z}'\mathbf{e}_N = \mathbf{Z}\mathbf{e}_M + \mathbf{y}\,, \qquad \mathbf{e}'_N \tilde{\mathbf{A}} \langle \mathbf{e}'_N \mathbf{Z} \rangle = \mathbf{e}'_N \mathbf{Z} + \mathbf{v}'\,.$$

Proceeding by using formula (13) gives

$$\tilde{\mathbf{A}}\mathbf{Z}'\mathbf{e}_N = \tilde{\mathbf{G}}\mathbf{Z}'\mathbf{e}_N + \mathbf{y}\,, \qquad \langle \mathbf{e}'_N \tilde{\mathbf{A}} \rangle \mathbf{Z}'\mathbf{e}_N = \mathbf{Z}'\mathbf{e}_N + \mathbf{v}$$

and finally we get the system of equations

$$(\mathbf{X}_0 - \mathbf{Z}_0)\langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1} \mathbf{Z}'\mathbf{e}_N = \mathbf{y}\,, \qquad \langle \mathbf{e}'_N (\mathbf{X}_0 - \mathbf{Z}_0) \rangle \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1} \mathbf{Z}'\mathbf{e}_N = \mathbf{v}$$

that exactly corresponds to the system (5), (6) provided that $\hat{\mathbf{q}} = \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1} \langle \mathbf{e}'_N \mathbf{Z} \rangle$.

It is important to emphasize here that using matrix-valued production function (12) with quasi-reciprocal technical coefficients leads to the same result as introducing matrix-valued cost function (11) with conventional (Leontief) coefficients. This fact can be appreciated as explicit testimony of dual nature of input-output modeling at constant prices.

5. Exploring an input-output model at constant production level

According to the ***first*** equation (7), the column vector of product outputs is equal to $\mathbf{X}\mathbf{e}_M = \hat{\mathbf{p}}\mathbf{X}_0\mathbf{e}_M = \langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{p}$ from which the price index **p** follows as $\hat{\mathbf{p}} = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}$ where the commutativity property of diagonal matrices is used again. Substituting the latter expression in multiplicative patterns (7) gives two linear matrix-valued functions

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \mathbf{H}\,, \tag{14}$$

$$\mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{Z}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \mathbf{B} \tag{15}$$

with vector of product outputs $\mathbf{X}\mathbf{e}_M$ as their common argument. Note that matrix $\mathbf{H} = \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}_0$ is known in special literature as market shares matrix (see Eurostat, 2008) with contributions of each industry to the output of a product in a row. The matrix **H** in (14) provides a linkage between production matrix **X** and its row marginal totals.

In turn, matrix-valued function (15) establishes a linear dependency of intermediate consumption matrix **Z** from the product outputs $\mathbf{X}\mathbf{e}_M$, and so it can be classified as *the linear matrix-valued cost function*. Matrix $\mathbf{B} = \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$ is known under the name of (Ghosh) allocation coefficients matrix (see, e.g., Miller and Blair, 2009).

Substituting the matrix-valued cost function (15) in the equations of input-output model (1), (2), we have

$$\mathbf{X}\mathbf{e}_M = \langle \mathbf{X}\mathbf{e}_M \rangle \mathbf{B}\mathbf{e}_M + \mathbf{y}, \qquad \mathbf{e}'_N \mathbf{X} = \mathbf{e}'_M \mathbf{X}'\mathbf{B} + \mathbf{v}'.$$

Proceeding by using formula (14) gives

$$\mathbf{X}\mathbf{e}_M = \langle \mathbf{B}\mathbf{e}_M \rangle \mathbf{X}\mathbf{e}_M + \mathbf{y}, \qquad \mathbf{H}'\mathbf{X}\mathbf{e}_M = \mathbf{B}'\mathbf{X}\mathbf{e}_M + \mathbf{v},$$

and finally we obtain system of equations

$$\langle (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M \rangle \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}\mathbf{e}_M = \mathbf{y}, \qquad (\mathbf{X}'_0 - \mathbf{Z}'_0)\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}\mathbf{e}_M = \mathbf{v}$$

that exactly corresponds to the system (8), (9) provided that $\hat{\mathbf{p}} = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}$.

From the other side, in accordance with the ***second*** equation (7) the column vector of product intermediate consumption is equal to $\mathbf{Z}\mathbf{e}_M = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{e}_M = \langle \mathbf{Z}_0\mathbf{e}_M \rangle \mathbf{p}$ from which index $\mathbf{p}$ follows as $\hat{\mathbf{p}} = \langle \mathbf{Z}\mathbf{e}_M \rangle \langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1}$. Substituting the latter expression in multiplicative patterns (7) yields two linear matrix-valued functions

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0 = \langle \mathbf{Z}\mathbf{e}_M \rangle \langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}_0 = \langle \mathbf{Z}\mathbf{e}_M \rangle \tilde{\mathbf{B}}, \qquad (16)$$

$$\mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0 = \langle \mathbf{Z}\mathbf{e}_M \rangle \langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1} \mathbf{Z}_0 = \langle \mathbf{Z}\mathbf{e}_M \rangle \tilde{\mathbf{H}} \qquad (17)$$

with vector of product intermediate inputs $\mathbf{Z}\mathbf{e}_M$ as their common argument. Here matrix-valued function (16) provides a linear dependency of output matrix $\mathbf{X}$ from the product intermediate inputs $\mathbf{Z}\mathbf{e}_M$, and so it can be classified as *the linear matrix-valued production function*. Matrix $\tilde{\mathbf{B}} = \langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}_0$ is apparently not known in special literature in contrast to Ghosh allocation coefficients matrix $\mathbf{B} = \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$, but it is easy to see that matrices $\tilde{\mathbf{B}}$ and $\mathbf{B}$ are in certain "quasi-reciprocal" relation.

The equation (17) establishes a linkage between intermediate consumption matrix $\mathbf{Z}$ and its row marginal totals. Matrix $\tilde{\mathbf{H}} = \langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$ are apparently not mention in special literature in contrast to its well-known twin – the market shares matrix $\mathbf{H} = \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1} \mathbf{X}_0$.

Substituting the matrix-valued production function (16) in the equations of input-output model (1), (2), we obtain

$$\langle \mathbf{Z}\mathbf{e}_M \rangle \tilde{\mathbf{B}}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y}, \qquad \mathbf{e}'_M \mathbf{Z}'\tilde{\mathbf{B}} = \mathbf{e}'_N \mathbf{Z} + \mathbf{v}'.$$

Proceeding by using formula (17) gives

$$\langle \tilde{\mathbf{B}}\mathbf{e}_M \rangle \mathbf{Z}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y}, \qquad \tilde{\mathbf{B}}'\mathbf{Z}\mathbf{e}_M = \tilde{\mathbf{H}}'\mathbf{Z}\mathbf{e}_M + \mathbf{v}$$

and finally we get the system of equations

$$\langle(\mathbf{X}_0-\mathbf{Z}_0)\,\mathbf{e}_M\rangle\langle\mathbf{Z}_0\mathbf{e}_M\rangle^{-1}\mathbf{Z}\mathbf{e}_M=\mathbf{y}\,,\qquad(\mathbf{X}'_0-\mathbf{Z}'_0)\langle\mathbf{Z}_0\mathbf{e}_M\rangle^{-1}\mathbf{Z}\mathbf{e}_M=\mathbf{v}$$

that exactly corresponds to the system (5), (6) provided that $\hat{\mathbf{p}}=\langle\mathbf{Z}\mathbf{e}_M\rangle\langle\mathbf{Z}_0\mathbf{e}_M\rangle^{-1}$.

Here it is worth to mention that using matrix-valued production function (16) with quasi-reciprocal allocation coefficients leads to the same result as introducing matrix-valued cost function (17) with conventional (Ghosh) coefficients. This fact can be appreciated as ostensive evidence of dual nature of input-output modeling at constant level of production.

6. Regular and supplementary solutions for the model at constant prices

Consider some operational opportunities in obtaining solutions for the input-output model (5), (6) in the cases of evaluating a response of the economy to exogenous changes in the net final demand vector $\mathbf{y}=\mathbf{y}_*\neq\mathbf{y}_0$ with dimensions $N\times1$ or in the value added vector $\mathbf{v}=\mathbf{v}_*\neq\mathbf{v}_0$ with dimensions $M\times1$ at constant prices. Here it is assumed that “disturbed” vectors $\mathbf{y}_*$ and $\mathbf{v}_*$ do not have any zero components.

The material balance model (5) contains N linear equations with M scalar variables $\mathbf{q}$, whereas the financial balance model (6) includes M linear equations with the same M unknowns. Hence, in most general case $N >=< M$ one can assess a response of the economy only to exogenous change in the value added vector $\mathbf{v}=\mathbf{v}_*\neq\mathbf{v}_0$ by resolving the equation (6) written as $\langle\mathbf{e}'_N(\mathbf{X}_0-\mathbf{Z}_0)\rangle\mathbf{q}=\hat{\mathbf{v}}_0\mathbf{q}=\mathbf{v}_*$ with respect to the column vector of the relative quantity indices for industries, namely

$$\mathbf{q}=\hat{\mathbf{v}}_0^{-1}\mathbf{v}_*\,. \tag{18}$$

It should be noted that the solution (18) is valid at any numbers of products and industries in the economy. Nevertheless, this regular solution is trivial because a response of input-output model (5), (6) to the disturbance $\mathbf{v}=\mathbf{v}_*\neq\mathbf{v}_0$ comes to the alternate multiplying the columns of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the growth indices of value added through all industries at constant prices on the products.

However, at $N=M=K$ a choice of alternative exogenous condition is also feasible in finding a supplementary solution for the model (5), (6). Under the exogenous final demand condition $\mathbf{y}=\mathbf{y}_*\neq\mathbf{y}_0$, the equation (5) written as $(\mathbf{X}_0-\mathbf{Z}_0)\,\mathbf{q}=\mathbf{y}_*$ can be resolved with respect to the column vector of the relative quantity indices for industries, namely

$$\mathbf{q}=(\mathbf{X}_0-\mathbf{Z}_0)^{-1}\mathbf{y}_*\,, \tag{19}$$

of course, if an inverse of the square (at $N = M = K$) matrix $\mathbf{X}_0-\mathbf{Z}_0$ exists as it is expected to be. (Note that initial production matrix $\mathbf{X}_0$ usually has the dominant main diagonal.) The supplementary solution (19) is valid only if the values of N and M coincide, but it is not trivial in contrast to regular solution (18).

7. Regular and supplementary solutions for the model at constant production level

In its turn, consider operational opportunities in getting solutions for the input-output model (8), (9) in the cases of evaluating a response of the economy to exogenous changes in the final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ or in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ at constant level of production.

The material balance model (8) contains N linear equations with N scalar variables $\mathbf{p}$, whereas the financial balance model (9) includes M linear equations with the same N unknowns. Hence, in a general case $N >=< M$ one can evaluate a response of the economy only to exogenous change in the final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ by resolving the equation (8) written as $\langle (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M \rangle \mathbf{p} = \hat{\mathbf{y}}_0\mathbf{p} = \mathbf{y}_*$ with respect to the column vector of the relative price indices on products, namely

$$\mathbf{p} = \hat{\mathbf{y}}_0^{-1}\mathbf{y}_* . \tag{20}$$

The regular solution (20) is valid at any numbers of products and industries in the economy. Nevertheless, this solution is trivial because a response of input-output model (8), (9) to the disturbance $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ comes to the alternate multiplying the rows of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the value indices of final demand through all products at constant level of production in the industries.

However, at $N = M = K$ a choice of alternative exogenous condition is also feasible in finding a supplementary solution for the model (8), (9). Under the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, the equation (9) written as $(\mathbf{X}'_0 - \mathbf{Z}'_0)\mathbf{p} = \mathbf{v}_*$ can be resolved with respect to the column vector of the relative price indices on products, namely

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* , \tag{21}$$

of course, if an inverse of the square (at $N = M = K$) matrix $\mathbf{X}'_0 - \mathbf{Z}'_0$ exists. (Recall that initial production matrix $\mathbf{X}_0$ usually has the dominant main diagonal.) The supplementary solution (21) is valid only if the values of N and M coincide, but in contrast to the regular solution (20), it is not trivial.

It is interesting here to pay attention to the fact that models (5), (6) and (8), (9) do demonstrate remarkable duality properties in pairwise comparison of the regular solutions (18) and (20) at any values N and M as well as the supplementary solutions (19) and (21) at $N = M = K$ respectively.

8. Generalized versions of Leontief demand-driven model and Ghosh supply-driven model

The model (5), (6) and its supplementary solution (19) together with the resulting disturbances in production and intermediate consumption matrices (4) describe an impact of exogenous changes in final demand in terms of the production quantity changing at constant prices on the products. The model (8), (9) and its supplementary solution (21) together with the resulting disturbances in production and intermediate consumption matrices (7) characterize an impact of exogenous changes in value added in terms of price changing at constant level of production in the industries.

Model (5), (6) at $N = M = K$ can be considered as a generalized version of well-known Leontief demand-driven model (see Miller and Blair, 2009, Section 2.2.2). It serves to assess an impact of exogenous (absolute or relative) changes in final demand on the economy at constant prices. Indeed, as it follows from (4), the main fundamentals of model (5), (6) are $\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}$ and $\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}$ where

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = [\langle \mathbf{X}_0\mathbf{e}_K\rangle(\mathbf{H} - \mathbf{B})]^{-1}\mathbf{y}_* = (\mathbf{H} - \mathbf{B})^{-1}\langle \mathbf{X}_0\mathbf{e}_K\rangle^{-1}\mathbf{y}_* \tag{22}$$

according to (19). Total requirements matrix, which links the vector of product outputs with the final demand vector, can be derived as follows:

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q} = \mathbf{X}_0(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{X}_0^{-1}\right]^{-1}\mathbf{y}_* = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\right)^{-1}\mathbf{y}_*. \tag{23}$$

Note that generalized technical coefficients $\mathbf{Z}_0\mathbf{X}_0^{-1}$ have been explored by Jansen and ten Raa (1990) and other authors in the context of constructing symmetric input-output tables; this form of technical coefficients is known as commodity technology model.

Model (8), (9) at $N = M = K$ can be classified as a generalized version of Ghosh supply-driven model (see Miller and Blair, 2009, Section 12.1). It helps to evaluate an impact of exogenous (absolute or relative) changes in value added on the economy at fixed production scales (at constant level of production). As it follows from (7), the main fundamentals of model (8), (9) are $\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0$ and $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0$ where

$$\mathbf{p} = (\mathbf{X}_0' - \mathbf{Z}_0')^{-1}\mathbf{v}_* = [\langle \mathbf{e}_K'\mathbf{X}_0\rangle(\mathbf{G}' - \mathbf{A}')]^{-1}\mathbf{v}_* = (\mathbf{G}' - \mathbf{A}')^{-1}\langle \mathbf{e}_K'\mathbf{X}_0\rangle^{-1}\mathbf{v}_* \tag{24}$$

in accordance with (21). A Ghosh analogue of total requirements matrix, which links the vector of industry outputs with the value added vector, can be derived as follows:

$$\mathbf{X}'\mathbf{e}_K = \mathbf{X}_0'\mathbf{p} = \mathbf{X}_0'(\mathbf{X}_0' - \mathbf{Z}_0')^{-1}\mathbf{v}_* = \left[(\mathbf{X}_0' - \mathbf{Z}_0')(\mathbf{X}_0')^{-1}\right]^{-1}\mathbf{v}_* = \left[\mathbf{E}_K - \mathbf{Z}_0'(\mathbf{X}_0')^{-1}\right]^{-1}\mathbf{v}_*. \tag{25}$$

Here it is worth to mention the duality properties of models (5), (6) and (8), (9) again, because a response of model (5), (6) to the disturbance of the final demand coefficients

$\langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{y}_*$ is described in the equation (22) in terms of matrices **H** and **B**, whereas a response of model (8), (9) to the disturbance of the value added coefficients $\langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{v}_*$ is represented in the equation (24) in terms of matrices **G** and **A**.

9. The Leontief and Ghosh models for symmetric input-output table

Vectors **q** and **p** are defined in previous Section under an assumption that all the matrices in (22) – (25) are square (at $N = M = K$). In addition, let the initial production matrix $\mathbf{X}_0$ be a diagonal one as in a symmetric input-output table. Then the generalized versions of Leontief and Ghosh models considered above can be easily led to a “classical” view.

Diagonal matrix $\mathbf{X}_0$ of order K has a following row of properties:

$$\mathbf{X}_0 = \mathbf{X}'_0 = \langle \mathbf{e}'_K \mathbf{X}_0 \rangle = \langle \mathbf{X}_0 \mathbf{e}_K \rangle . \qquad (26)$$

The most famous Leontief formula for demand-driven model can be obtained using (22), (26) and some algebraic properties of diagonal matrices along the sequential transformations of the product outputs vector $\mathbf{X}\mathbf{e}_K$ as follows:

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}_0 \mathbf{q} = \mathbf{X}_0 (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0) \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1} \right]^{-1} \mathbf{y}_* = (\mathbf{E}_K - \mathbf{A})^{-1} \mathbf{y}_* \qquad (27)$$

where **A** is the (Leontief) technical coefficients matrix, as earlier.

Its analogue for Ghosh supply-driven model can be easily derived in the similar manner, using (24) and then (26) along the sequential transformations of the industry outputs vector $\mathbf{X}'\mathbf{e}_K$ as follows:

$$\mathbf{X}'\mathbf{e}_K = \mathbf{X}'_0 \mathbf{p} = \mathbf{X}'_0 (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1} \mathbf{v}_* = \left[(\mathbf{X}'_0 - \mathbf{Z}'_0) \langle \mathbf{X}'_0 \mathbf{e}_K \rangle^{-1} \right]^{-1} \mathbf{v}_* = (\mathbf{E}_K - \mathbf{B}')^{-1} \mathbf{v}_* \qquad (28)$$

where **B** is the (Ghosh) allocation coefficients matrix, as earlier.

It is to be emphasized that direct putting (26) into the main statement for generalized version of Ghosh supply-driven model (24) gives well-known formula

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1} \mathbf{v}_* = (\langle \mathbf{e}'_K \mathbf{X}_0 \rangle - \mathbf{Z}'_0)^{-1} \mathbf{v}_* = (\mathbf{E}_K - \mathbf{A}')^{-1} \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1} \mathbf{v}_* \qquad (29)$$

for so-called Leontief price model (see Miller and Blair, 2009, p. 44). Thus, in the case of a symmetric input-output table (when $\mathbf{X}_0$ is diagonal matrix) the Ghosh supply-driven model is equivalent to the Leontief price model (see also Dietzenbacher, 1997, for more details about interrelation of these models).

It can be shown in similar manner that the Leontief demand-driven model serves as the Ghosh quantity model. Indeed, direct substituting (26) into the main statement for generalized version of Leontief demand-driven model (22) gives a brief proof of this fact, namely

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_* = (\langle \mathbf{X}_0 \mathbf{e}_K \rangle - \mathbf{Z}_0)^{-1} \mathbf{y}_* = (\mathbf{E}_K - \mathbf{B})^{-1} \langle \mathbf{X}_0 \mathbf{e}_K \rangle^{-1} \mathbf{y}_* . \qquad (30)$$

It is appropriate to mention that all formulas obtained above in this and previous Section demonstrate a remarkable set of duality properties, especially in pairwise comparisons. For instance, generalized version of the technical coefficients matrix $\mathbf{A} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ in (27) is matrix $\mathbf{Z}_0 \mathbf{X}_0^{-1}$ in (23), whereas generalized version of the allocation coefficients matrix $\mathbf{B} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$ in (28) is matrix $\mathbf{X}_0^{-1} \mathbf{Z}_0$ in (25).

10. Concluding remarks

A general formulation of linear input-output model is considered in the paper as a system of equations written in terms of free variables for any rectangular input-output (or supply and use) table given. This system spans the regular linear equations for material and financial balances with a batch of predetermined values for exogenous variables (final demand and value added vectors).

Any variations in exogenous elements of input–output model lead to the changes of price and quantity proportions in the resulting supply and use table that are formally described by the nonlinear multiplicative patterns (3). These patterns can be adjusted for evaluating the input–output model at constant prices in linear form (4) and at constant level of production in linear form (7).

The proposed approach for assessing the model at constant prices provides an exact identifiability of the model within rectangular and square formats by introducing to the system of its equations either the linear matrix-valued cost function with industry outputs as its arguments based on (Leontief) technical coefficients (11) or the linear matrix-valued production function with industry inputs as its arguments based on industry productive (quasi-reciprocal technical) coefficients (12). In contrast, the model at constant level of production is exactly identifiable provided that one involves in it either the linear matrix-valued cost function with product outputs as its arguments based on (Ghosh) allocation coefficients (15) or the linear matrix-valued production function with product inputs as its arguments based on product multiplication (quasi-reciprocal allocation) coefficients (16).

It is important to emphasize that in a case of constant prices using matrix-valued production function (12) with quasi-reciprocal technical coefficients leads to the same result as introducing matrix-valued cost function (11) with conventional Leontief coefficients.

Analogously, in a case of constant production level results of using cost function with Ghosh coefficients (15) and production function with quasi-reciprocal allocation coefficients (16) also coincide between each other. Thus, technical and allocation coefficients should be regarded as helpful ways of economic interpretation rather than as basic framework or operational tools for modeling, contrary to a widely accepted point of view that “the center-piece of input–output analysis is a matrix... of technical coefficients” (ten Raa, 1994, p.4). Moreover, equivalence of the models with the matrix-valued production functions and the models with the matrix-valued cost functions can be appreciated as a clear demonstration of general equilibrium in the theory of input-output analysis and an ostensive evidence of dual nature of input-output modeling.

The regular (at any values N and M) and supplementary (at $N = M$) solutions for model (5), (6) at constant prices are derived in (18), (19) and for model (8), (9) at constant production level are obtained in (20), (21). Square models (5), (6) and (8), (9) with the supplementary solutions (19) and (21) can be classified as generalized versions of Leontief demand-driven model and Ghosh supply-driven model respectively.

In a case of symmetric input-output table, the properties of diagonal production matrix allow transforming the generalized versions of Leontief and Ghosh models into the “classical” input–output models. In this context, the equivalence of Leontief price model and Ghosh supply-driven model as well as the equivalence of Leontief demand-driven model and Ghosh quantity model is proven. It is interesting to note that relevant formulas do demonstrate a remarkable set of duality properties.

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